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Solution of the Laplace inversion problem for a special function

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§ 1. Introduction

A problem in the theory of electromagnetic waves studied by B. van der Pol [1] led to the question: Does there exist a function h(t) so that

$$(1) \qquad f(p) = \int_0^\infty \frac{e^{-z\sqrt{x^2+a^2p^2}}\, J_0(\rho x)\, x\, dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}$$

is the Laplace transform of h(t) in the sense that

$$(2) \qquad f(p) = p\int_0^\infty e^{-pt}\, h(t)\, dt \ ?$$

And if the answer is affirmative, give a manageable expression for this function h(t).

These problems will be solved in this paper by means of the complex inversion theorem for Laplace transforms ([2], Satz 21.2, p. 182). However, this theorem cannot be applied to f(p). Therefore, in § 2, we shall study a function $f_\mu(p)$, to which the inversion theorem applies if $\mu > 0$, and which has the property $f_\mu(p) \to f(p)$ if $\mu \to 0$. We shall find a function $h_\mu(t)$ which is related to $f_\mu(p)$ by (2) if $\mu > 0$. In § 3 we prove that $h_\mu(t)$ has a limit h(t) if $\mu \to 0$, and in § 4 it will be shown that this function h(t) solves our problem.

Finally, in § 5 we shall give the required manageable expressions, namely complete elliptic integrals.

Throughout the paper it will be assumed that ρ, z, a, b, c, d are positive numbers, and $a \neq b$.

The method of this paper applies equally well if in (1) $J_0(\rho x)$ is replaced by $J_\nu(\rho x)$, where ν is a natural number.

§ 2. A generalization.

In this section we consider the function

$$(3) \qquad f_\mu(p) = \int_0^\infty \frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} J_0(\rho x)\, x\, dx$$

for positive values of μ. (If $\mu=0$ we have again the function $f(p)$ defined in the previous section.) We shall try to find the original $h_\mu(t)$ of $f_\mu(p)$ in the sense of (2). As we intend to apply the complex inversion formula for Laplace transforms, we have to investigate the analytic continuation of $f_\mu(p)$ into a right half-plane. Therefore it is necessary to define the functions $\sqrt{x^2+a^2p^2}$ and $\sqrt{x^2+b^2p^2}$ for complex values of p. We make two cuts C_a and C_b in the complex w-plane. C_a consists of the two intervals $(\frac{i}{a}, i\infty)$ and $(-\frac{i}{a}, -i\infty)$ on the imaginary axis. $\sqrt{1+a^2w^2}$ is defined in the w-plane with cut C_a so that the root is positive on the real axis. If $p=xw$ then $\sqrt{x^2+a^2p^2}$ is defined as $x\sqrt{1+a^2w^2}$. In an analogous way C_b and $\sqrt{x^2+b^2p^2}$ are defined. It is not difficult to prove that

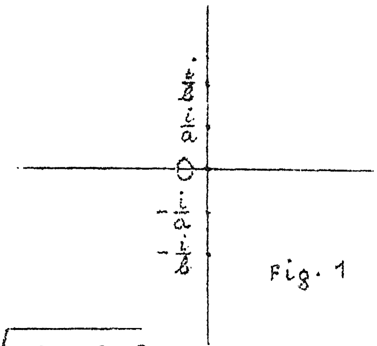

Fig. 1

$$(4) \qquad \mathrm{Re}\sqrt{1+a^2w^2} \geqslant \mathrm{Re}\, a w.$$

Applying this in the case $\mathrm{Re}\, p > 0$, we find

$$\mathrm{Re}\sqrt{x^2+a^2p^2} \geqslant 0, \qquad \mathrm{Re}\sqrt{x^2+b^2p^2} \geqslant 0.$$

If $p=\sigma+i\tau$ $(\sigma>0,\ \tau>0)$, then we have

$$\mathrm{Im}\sqrt{x^2+a^2p^2} > 0,$$

and

$$\left|\sqrt{x^2+a^2p^2}\right| = \sqrt{\left|x^2+a^2(\sigma^2-\tau^2)+2a^2\sigma\tau i\right|} \geqslant a\sqrt{2\sigma\tau}.$$

And in a similar way

$$\mathrm{Im}\sqrt{x^2+b^2p^2} > 0, \quad \left|\sqrt{x^2+b^2p^2}\right| \geqslant b\sqrt{2\nu\tau}.$$

Therefore we find

$$\left|c\sqrt{x^2+a^2p^2} + d\sqrt{x^2+b^2p^2}\right| \geqslant \sqrt{2\sigma|\tau|}\sqrt{a^2c^2+b^2d^2}.$$

The same result holds if $\tau<0$.

If $|\tau| < \sigma$ we also have the estimations

$$\left|\sqrt{x^2+a^2p^2}\right| \geq a\sqrt{\sigma^2-\tau^2}, \quad \left|\sqrt{x^2+b^2p^2}\right| \geq b\sqrt{\sigma^2-\tau^2},$$

$$\left|c\sqrt{x^2+a^2p^2} + d\sqrt{x^2+b^2p^2}\right| \geq \sqrt{\sigma^2-\tau^2}\sqrt{a^2c^2+b^2d^2} .$$

Applying these results we find

$$(5) \qquad \left|\frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}} J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}\right| \leq \gamma \frac{e^{-\mu x}|J_0(\rho x)|x}{\sqrt{a^2c^2+b^2d^2}} ,$$

where $\gamma = (2\sigma|\tau|)^{-\frac{1}{2}}$, and if $|\tau| < \sigma$ we may also take $\gamma = (\sigma^2-\tau^2)^{-\frac{1}{2}}$. It follows from this that the integral (3) is absolutely convergent in the half plane Re $p > 0$, and that

$$|f_\mu(p)| \to 0 \quad \text{if} \quad |p| \to \infty$$

uniformly in the halfplane Re $p \geq \beta$ (β is an arbitrary positive number).

Another consequence of (5) is

$$\int_{\alpha - i\infty}^{\alpha + i\infty} \left|\frac{f_\mu(p)}{p}\right| dp < \infty$$

if $\alpha > 0$.

Finally, it can be shown that $f_\mu(p)$ is an analytic function in the half plane Re $p > 0$.

$f_\mu(p)$ satisfies the conditions of the complex inversion theorem ([2] , Satz 21.2, p.182). Hence,the function $h_\mu(t)$ defined by

$$(6) \qquad h_\mu(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} \frac{e^{pt}}{p} f_\mu(p)dp \qquad\qquad (\alpha > 0)$$

equals 0 if $t < 0$ and $f_\mu(p)$ is the Laplace transform of $h_\mu(t)$.

Next in (6) we substitute the integral expression (3) for $f_\mu(p)$ and interchange the order of integration. This procedure can be justified in the following way. If Re $p = \alpha$, it follows from (5) that

$$g(p) = \int_0^\infty \left|\frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}} J_0(x\rho)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}\right| dx \leq \begin{cases} \dfrac{C}{\sqrt{|\tau|}} & \text{if } |\tau| \geq \alpha \\[2mm] \dfrac{C}{\sqrt{\alpha^2-\tau^2}} & \text{if } |\tau| < \alpha, \end{cases}$$

where C does not depend on τ . Therefore

$$\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{pt}}{p}\, g(p)\, dp$$

converges absolutely, and we have

$$(7)\qquad h_\mu(t)=\int_0^\infty J_o(\rho x)dx\, \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{pt-\mu x-z\sqrt{x^2+a^2p^2}}\, x\, dp}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}\, p} =$$

$$\int_0^\infty J_o(\rho x)dx\, \frac{1}{2\pi i}\int_{\frac{\alpha}{x}-i\infty}^{\frac{\alpha}{x}+i\infty} \frac{e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}}{c\sqrt{1+a^2w^2}+d\sqrt{1+b^2w^2}}\, \frac{dw}{w}\,.$$

It is easily seen that

$$(8)\qquad \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}}{c\sqrt{1+a^2w^2}+d\sqrt{1+b^2w^2}}\, \frac{dw}{w}$$

is independent of β, as long as $\beta > 0$. For, if $0<\beta_1 \leq \text{Re } w \leq \beta_2$, then

$$e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}$$

is a bounded function of w. This follows from the estimate

$$\text{Re}(\mu+z\sqrt{1+a^2w^2}-wt)x \geqslant \left\{\mu + (az-t)\text{Re } w\right\} x.$$

Another consequence of this inequality is that $e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}$ is a bounded function of w in the half plane Re $w>0$ if $t \leq az$. Therefore, the integral (8) and hence $h_\mu(t)$ equals zero in this case. From now on we assume $t > az$. We have deduced that, if $\beta > 0$,

$$(9)\qquad h_\mu(t) = \int_0^\infty J_o(\rho x)dx\, \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{e^{-(\mu+z\sqrt{1+a^2w^2}-wt)x}}{c\sqrt{1+a^2w^2}+d\sqrt{1+b^2w^2}}\, \frac{dw}{w}\,.$$

We again want to change the order of the integrations. This can easily be justified, if

$$0 < \beta < \frac{\mu}{t-az}\,.$$

In that case we have

$$\left| \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{e^{-(\mu + z\sqrt{1+a^2w^2} - wt)x}}{c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}} \frac{dw}{w} \right| \leq e^{-(\mu + (az-t)\beta)x} \, C ,$$

where

$$C = \frac{1}{2\pi} \int_{\beta - i\infty}^{\beta + i\infty} \frac{|dw|}{|c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}|\,|w|}$$

is independent of x. As $\mu + (az-t)\beta > 0$, the integral

$$\int_0^\infty J_0(\rho x) e^{-(\mu + (az-t)\beta)x} \, dx$$

converges absolutely. Hence, we have proved

$$h_\mu(t) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{dw}{w\{c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}\}}.$$

(10)
$$\int_0^\infty J_0(x\rho) e^{-(\mu + z\sqrt{1+a^2w^2} - wt)x} \, dx =$$

$$= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{dw}{w\{c\sqrt{1+a^2w^2} + d\sqrt{1+b^2w^2}\}\sqrt{\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2}}$$

Here $\sqrt{\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2}$ must be taken positive if $w = \beta$ ([3] ,p.47).

$\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2$ can be factorized into

$$(z\sqrt{1+a^2w^2} - wt + \lambda)(z\sqrt{1+a^2w^2} - wt + \bar{\lambda}) \qquad (\lambda = \mu + i\rho).$$

Each factor has only one zero in the w-plane with cut C_a. These zeros w_1 en $\overline{w_1}$ have real parts $\geq \frac{\mu}{t-az} > \beta$. Hence, we can replace the integration contour Re $w = \beta$ by the contour W, which is shown in fig. 2. In A we have to take $\sqrt{\rho^2 + (\mu + z\sqrt{1+a^2w^2} - wt)^2}$ positive.

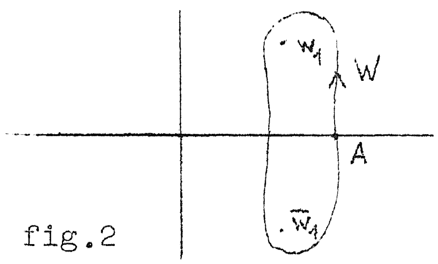

fig.2

Another integral representation of the function $h_\mu(t)$ is obtained by applying the conformal mapping

(11)
$$u = \frac{\sqrt{1+a^2w^2}}{w} .$$

The cut C_a is mapped onto the interval $(-a,a)$. As to the cut C_b we have to distinguish the two cases: I. $a < b$ and II. $a > b$. In fig. 3 and fig.4 the cuts for the integrand and the integration contours V_1 and V_2 are sketched. u_1 and $\overline{u_1}$ are the images of w_1 and $\overline{w_1}$. A' is the image of A.

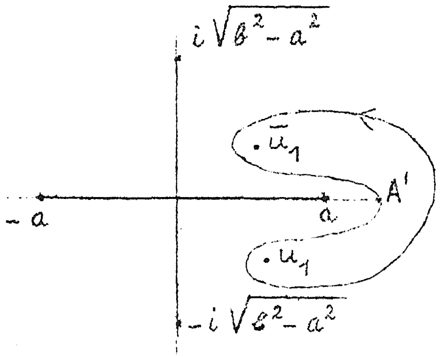

fig.3

I. $a < b$

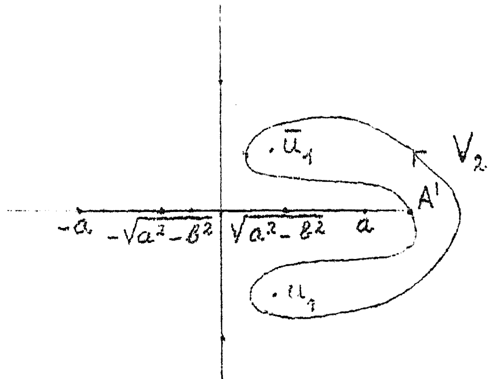

fig.4

II. $a > b$

In this way we find, if $j=1,2$,

$$(12)\quad h_{\mu}(t) = -\frac{1}{2\pi i}\int_{V_j} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}},$$

where in A', $\sqrt{u^2-a^2}$, $\sqrt{u^2+b^2-a^2}$ and $\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}$ are positive.

Finally, if $\mu \to 0$ we can derive in the case $t > Ra$, where

$$R = \sqrt{\rho^2+z^2}$$

that u_1 and $\overline{u_1}$ tend to

$$\frac{zt \pm i\rho\sqrt{t^2-a^2R^2}}{R^2},$$

whereas in the case $t < Ra$ u_1 and $\overline{u_1}$ tend to the same point

$$\frac{zt+\rho\sqrt{a^2R^2-t^2}}{R^2}.$$

§ 3. The limit case $\mu \to 0$.

In accordance with the method explained in § 1, we shall try to extend the results of the last section to the limit case $\mu \to 0$. It will be proved here, that $f_\mu(p)$ and $h_\mu(t)$ have limits if $\mu \to 0$ $(\mu > 0)$.

First of all, if $p > 0$ then $f_\mu(p) \to f(p)$ $(\mu \to 0)$. This follows from Lebesgue's theorem on majorized convergence, for we have

$$\frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \to \frac{e^{-z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}$$

if $\mu \to 0$, and

$$(15)\qquad \left|\frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}}\right| \leq \frac{e^{-zx}|J_0(\rho x)|\,x}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} .$$

The function on the right of (15) is integrable over $(0,\infty)$. So the conditions of Lebesgue's theorem are satisfied and we have

$$f_\mu(p) = \int_0^\infty \frac{e^{-\mu x - z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} \to \int_0^\infty \frac{e^{-z\sqrt{x^2+a^2p^2}}\,J_0(\rho x)x\,dx}{c\sqrt{x^2+a^2p^2}+d\sqrt{x^2+b^2p^2}} = f(p).$$

In the following we consider $\lim h_\mu(t)$ in the two cases I. $a < b$ and II. $a > b$.

I. We take the integration contour V_1 of § 2 fig.3. u_1 and $\overline{u_1}$ are complex continuous functions of μ $(\mu \geq 0)$, which assume real values only in the case $t < Ra$, $\mu = 0$, and take never purely imaginary values.

It is easily seen that $h_\mu(t)$ depends continuously on μ $(\mu \geq 0)$ in those points μ_0 where u_1 and $\overline{u_1}$ are not real. For we can take $\delta > 0$ so small that the sets

$$S = \{u_1(\mu)\,\big|\,|\mu - \mu_0| \leq \delta\} \quad\text{and}\quad T = \{\overline{u_1}(\mu)\,\big|\,|\mu - \mu_0| \leq \delta\},$$

do not contain for any μ with $|\mu - \mu_0| \leq \delta$ other singularities of the integrand $k_\mu(u,t)$ of (12) than $u_1(\mu)$, $\overline{u_1}(\mu)$ and we may take V_1 such that S and T are entirely inside V_1. Further, the integrand $k_\mu(u,t)$ tends uniformly to the limit

(16) $$\frac{u}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}}$$

if $u \in V_1$. So we have proved

(17) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_1} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $t > Ra$.

Next we suppose $t < Ra$. The foregoing considerations can be extended to the Riemann surface of $k_\mu(u,t)$. If $\mu \to 0$, then $u_1(\mu)$ and $\overline{u_1}(\mu)$ tend to points over the same point u_1 of the u-plane. The sets S and T on this Riemann surface are defined as in the case $t > Ra$. V_1^* (fig.5) is a simple contour on the Riemann surface which encircles S and T in the positive direction such that for all μ with $|\mu - \mu_0| \leq \delta$ the only singularities of $k_\mu(u,t)$ in the domain with boundary V_1^* are those in the sets S and T. If μ satisfies $0 < \mu \leq \delta$ we can deform V_1^* into a contour V_1 of the type described above without changing the value of the integral. It is also true that $k_\mu(u,t)$ tends to (16) uniformly on V_1^* if $\mu \downarrow 0$. Hence we may conclude

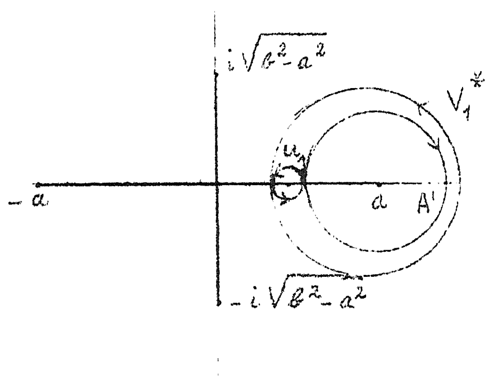

fig. 5

(18) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_1^*} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}}$$

if $\mu \downarrow 0$.

II. In an analogous way we can prove the existence of $\lim_{\mu \to 0} h_\mu(t)$ if $a > b$. We shall confine ourselves to a description of the limit function. Using now the integration contour V_2 of § 2 fig.4, we can deduce

(19) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_2} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $t > Ra$.

If $t < Ra$ we have to take care of the singularity $+\sqrt{a^2-b^2}$ of $k_\mu(u,t)$.

$u_1 = \dfrac{zt+\rho\sqrt{a^2R^2-t^2}}{R^2}$ is the greater root of

(19a) $$\rho^2(u^2-a^2) + (uz-t)^2 = 0.$$

Fig.6 is a u,t-diagram of this equation. It is an ellips with center in the origin. For the further discussion it is of interest to know whether $u_1 < \sqrt{a^2-b^2}$ or $u_1 > \sqrt{a^2-b^2}$. From the picture it is easily seen that $u_1 > \sqrt{a^2-b^2}$ if $\frac{za}{R} > \sqrt{a^2-b^2}$, that is if $Rb > \rho a$. However, if $Rb < \rho a$, it is also possible that $u_1 > \sqrt{a^2-b^2}$. Solving t from (19a) we find the condition

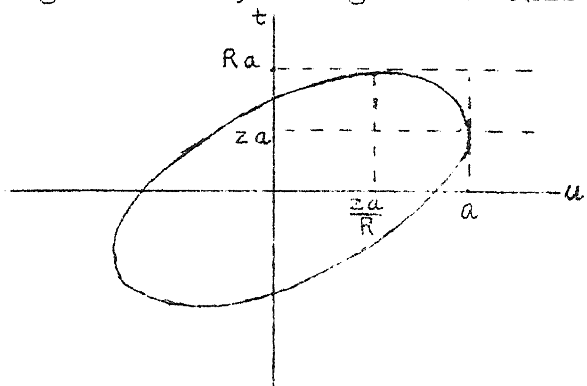

fig. 6

$t < z\sqrt{a^2-b^2}+\rho b$. Finally, $u_1 < \sqrt{a^2-b^2}$ only if $Rb < \rho a$ and $t > z\sqrt{a^2-b^2}+\rho b$. As in the case I we take a closed contour V_2^* if $u_1 > \sqrt{a^2-b^2}$ and a closed contour V_2^{**} if $u_1 < \sqrt{a^2-b^2}$ (fig.7a and 7b), and we find

(20) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_2^*} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $Rb > \rho a$ or $Rb < \rho a$ and $t < z\sqrt{a^2-b^2}+\rho b$.

(21) $$h_\mu(t) \to \frac{-1}{2\pi i}\int_{V_2^{**}} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(uz-t)^2}} \quad (\mu \downarrow 0)$$

if $Rb < \rho a$ and $t > z\sqrt{a^2-b^2}+\rho b$.

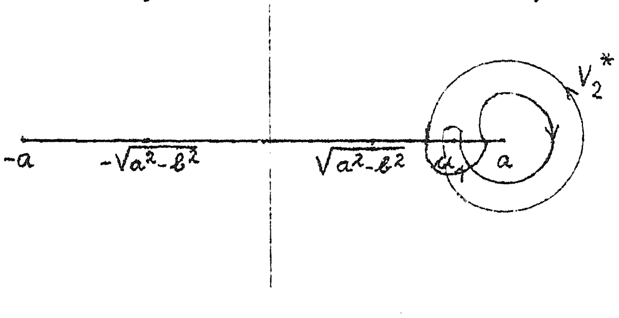

fig. 7a

$Rb > \rho a$ or $Rb < \rho a$ and $t < z\sqrt{a^2-b^2}+\rho b$

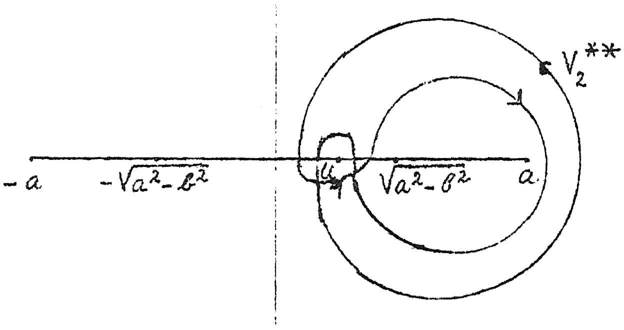

fig. 7b

$Rb < \rho a$ and $t > z\sqrt{a^2-b^2}+\rho b$

§ 4. Justification of the passage to the limit $\mu \to 0$

In the last section it was shown that

$$\lim_{\mu \downarrow 0} h_\mu(t) = h(t)$$

exists for all but a finite number of values of t. From the same section we know that

$$\lim_{\mu \downarrow 0} f_\mu(p) = f(p)$$

exists if $p > 0$. As we have

$$p \int_0^\infty h_\mu(t) e^{-pt} dt = f_\mu(p)$$

if $\mu > 0$, it is natural to expect that this equality holds even in the limit $\mu \to 0$. This can be proved by applying Lebesgue's theorem on majorized convergence. The conditions of this theorem are satisfied if a function g(t) exists such that

$$| h_\mu(t) | \leq g(t),$$

and

$$\int_0^\infty g(t) e^{-pt} dt < \infty \qquad (p > 0).$$

Such a function g(t) will be given here for the two cases I. $b > a$ and II. $b < a$.

I. If $b > a$ we can deform the integration contour V_1 of § 2, fig.3 into V_1' as is shown in fig.8. Taking into account the residu at $u=\infty$ we find

$$(22) \quad h_\mu(t) = \frac{1}{(c+d)\sqrt{\rho^2+(\mu+z)^2}} -$$

$$\frac{1}{2\pi} \int_{V_1'} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}}$$

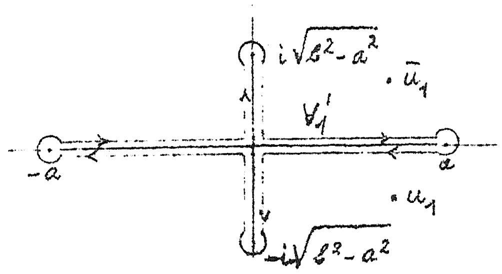

fig.8

As $c > 0$ and $d > 0$ we can show that by our definition of $\sqrt{u^2+b^2-a^2}$ $\psi(u)=cu+d\sqrt{u^2+b^2-a^2}$ has no zeros in the u-plane. Furthermore, $|\psi(u)|$ tends to infinity as $|u|\to\infty$. So we have $|\psi(u)| \geq k$ for some $k > 0$.

Now the following inequalities hold

$$|h_\mu(t)| \leq \frac{1}{(c+d)R} + \frac{1}{2\pi k} \int_{V_1'} \frac{|u| \; |du|}{\sqrt{|\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+\mu z-t)^2|}}$$

$$\text{(23)} \qquad \frac{1}{(c+d)R} + \frac{1}{\pi k}\int_{-a}^{a} \frac{|x| \, dx}{\sqrt{|\rho^2(x^2-a^2)+(\pm i\mu\sqrt{a^2-x^2}+xz-t)^2|}} + \frac{1}{\pi k}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \frac{|x| \, dx}{\sqrt{|-\rho^2(x^2+a^2)+(\pm i\mu\sqrt{x^2+a^2}+ixz-t)^2|}} .$$

If α, β, γ are real numbers, then

$$|-\alpha^2+(i\beta+\gamma)^2| \geq |-\alpha^2+\gamma^2| .$$

Using this inequality we find

$$|h_\mu(t)| \leq \frac{1}{(c+d)R} + \frac{1}{\pi k}\int_{-a}^{a} \frac{|x| \, dx}{\sqrt{|\rho^2(x^2-a^2)+(xz-t)^2|}} + \frac{1}{\pi k}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \frac{|x| dx}{\sqrt{|-\rho^2(x^2+a^2)+t^2|}} .$$

The function on the right of (24) can be taken as a majorizing $g(t)$. It depends continuously on t, except at the points $t=Ra$ and $t=\rho a$, where the quadratic expressions in x in the first and the second integral respectively have coinciding zeros. But in this points we can give the estimations $O(\log|t^2-R^2a^2|)$ and $O(\log|t^2-\rho^2a^2|)$ respectively. It is easily seen that $g(t)$ is bounded if $t\to\infty$.

II. If $b < a$ we deform the integration contour V_2 of § 2, fig.4 into V_2' (see fig.9).

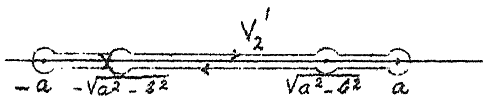

fig.9

Again we have a residu in $u=\infty$ and we find

$$h_\mu(t) = \frac{1}{(c+d)\sqrt{\rho^2+(\mu+z)^2}}$$

$$- \frac{1}{2\pi i} \int_{V_2} \frac{u\,du}{(cu+d\sqrt{u^2+b^2-a^2})\sqrt{\rho^2(u^2-a^2)+(\mu\sqrt{u^2-a^2}+uz-t)^2}} .$$

Proceeding as in the case I we obtain

$$(25) \quad |h_\mu(t)| \leq \frac{1}{(c+d)R} + \frac{1}{\pi k} \int_{-a}^{a} \frac{dx}{\sqrt{|(-\rho^2(a^2-x^2)+(xz-t)|^2}} .$$

The function at the right of (25) is continuous except at the point $t=Ra$, where the estimation $O(\log|R^2a^2-t^2|)$ holds. The function is also bounded if $t \to \infty$.

§ 5. Summary of the results.

It is possible to put the solution $h(t)$ of our problem in the form of complete elliptic integrals over intervals of the real axis. This can easily be done if we start from the formulae deduced in § 3. We again distinguish the two cases I and II.

I. If $a < b$ and $t > Ra$ we use (17). We deform the integration contour V_1 into the contour shown in fig.10, taking into account the residue at $u = \infty$. After some calculations we find

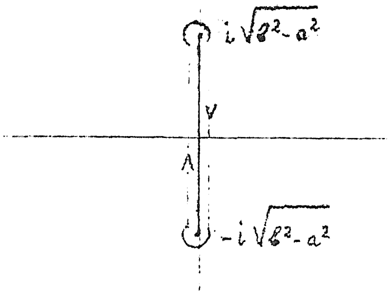

fig.10

(26) $$h(t) = + \frac{1}{(c+d)R} - \frac{1}{\pi i}\int_{-\sqrt{b^2-a^2}}^{\sqrt{b^2-a^2}} \frac{dx\sqrt{b^2-a^2-x^2}dx}{\{(c^2-d^2)x^2+d^2(b^2-a^2)\}\sqrt{-R^2x^2+2iztx+t^2-\rho^2a^2}},$$

where the roots are positive if $x=0$. It is not difficult to see that $h(t)$ assumes only real values.

If $t < Ra$ it is seen from (18) and fig.5 that V_1^* can be shrunk to the point u_1 and so $h(t)=0$ in this case.

II. When $a > b$, we consider first the case $t > Ra$. We deform the integration contour V_2 in (19) into the contour shown in fig.11. Proceeding as in the case $a < b$ we find

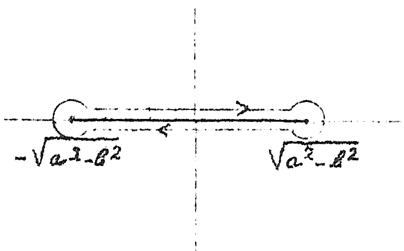

fig.11

(27) $$h(t) = + \frac{1}{(c+d)R} + + \frac{1}{\pi}\int_{-\sqrt{a^2-b^2}}^{\sqrt{a^2-b^2}} \frac{x\ d\sqrt{a^2-b^2-x^2}dx}{\{(c^2-d^2)x^2+d^2(a^2-b^2)\}\sqrt{R^2x^2-2ztx+t^2-\rho^2a^2}},$$

where the roots are non-negative.

Next we consider $Rb < \rho a$ and $z\sqrt{a^2-b^2}+\rho b < t < Ra$. It can be seen that V_2^{**} in fig.7b may be replaced by the contour of fig.12.

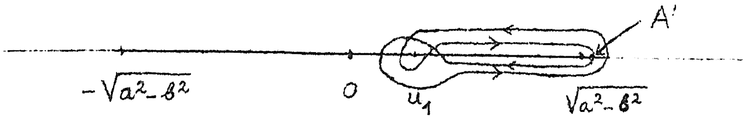

fig. 12

Therefore we find in this case

$$(28)\quad h(t) = +\frac{2}{\pi}\int_{u_1}^{\sqrt{a^2-b^2}} \frac{x\, d\sqrt{a^2-b^2-x^2}\,dx}{\{(c^2-d^2)x^2+d^2(a^2-b^2)\}\sqrt{R^2x^2-2ztx+t^2-\rho^2a^2}},$$

where $u_1 = \frac{zt+\rho\sqrt{R^2a^2-t^2}}{R^2}$ and the roots are non-negative.

Finally, if $Rb > \rho a$ or $Rb < \rho a$ and $t < z\sqrt{a^2-b^2}+\rho b$ we use (20). The contour V_2^* of fig.7a can again be shrunk to u_1. Therefore we find $h(t)=0$.

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